Fourier frames on measures with Fourier decay

Bochen Liu

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We say an object admits Fourier frames if its natural measure does.

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so, if $1 - |x'|^2 \approx 1$ on $\pi(C)$, then $\Lambda \times \{0\}$ is a frame spectrum for $C \subset S_+^{d-1}$ whenever Λ is a frame spectrum for $\pi(C) \subset \mathbb{R}^{d-1}$.

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Kolountzakis, Lai, 2025+: examples without tight frames (A=B).

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Chen, B.L., 2025+: generalize Iosevich-Lai-B.L.-Wyman (2022) to self-intersecting surfaces. In particular for planar curves we improve a result of Kolountzakis and Lai (2025+) from tight frames to frames.

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- Diophantine approximations: Kaufman-type construction (1981), still the only way to construct deterministic Salem sets in $\mathbb{R}$.

$\mathbb{R}^d, d \geq 2$: surfaces, random images and Diophantine approximations.

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In fact we prove a lot more, that is, such examples are "generic".

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We need a new criterion, especially on measures with Fourier decay.

A new criterion on measures with Fourier decay

Suppose Λ is a frame spectrum of μ.

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Contradiction if $t > s$! But does such a measure exist? Usually we work with Frostman measures $\mu(B(x,r)) \lesssim_\epsilon r^{\dim_{\mathcal{H}}(\operatorname{supp}\mu)-\epsilon}$, while $\dim_{\mathcal{F}} \leq \dim_{\mathcal{H}}$.

Heavy intervals in random Cantor sets

Li, B.L., 2024+: there exist s-dimensional Salem measures on Diophantine approximation such that for a rapidly increasing q_n,

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$s/2$ is optimal: $|\hat{\mu}(\xi)| \lesssim |\xi|^{-\frac{s}{2}} \implies \mu(B(x, r)) \lesssim r^{\frac{s}{2}}$ (Mitsis, 2002).

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Suppose $s > 0$ and μ is a Borel measure on $[0, 1]$ with

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Because of $\mu(B(x, r)) \lesssim r^s$, it is hard for μ_ω to have heavy intervals!

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Theorem (Li, B.L., 2025+)

Suppose $s > 0$ and μ is a Borel measure on $[0, 1]$ satisfying

$$\mu([0, r]) \lesssim r^{s/2},\ \forall r > 0,\ \textit{and}$$

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Diophantine approximation

Kaufman, 1981: Let q_i be a rapidly increasing sequence, then

$$\bigcap_i \bigcup_{H \in \{1,2\ldots,[q_i^{s/2}]\}} \mathcal{N}_{q_i^{-1}}\left(\frac{\mathbb{Z}}{H}\right) \cap [-\frac{1}{2}, \frac{1}{2}] \tag{1}$$

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We are running out of criteria...

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The key in the proof on the whole sphere is the surface measure

$$\widehat{\sigma}(\xi) = C\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\frac{d-1}{2}}\cos\left(2\pi\left(|\xi| - \frac{d-1}{8}\right)\right) + O(|\xi|^{-\frac{d-1}{2}-1}).$$

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$$R^{-d}\int_{|\xi|<R}|\hat{\sigma}(\lambda+\xi)|^2\,d\xi \approx |\lambda|^{-(d-1)}, \tag{2}$$

uniformly in $|\lambda| > 1$ and $1 < R < \frac{|\lambda|}{2}$, is what used in the proof.

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I do not know how to construct such a nice measure with (2) in $[0,1]$.

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and for each i and all integers $|k| > 2q_i, |l| < q_i/2$,

$$|\hat{\nu}(k + l)| \leq C\hat{\mu}(k) + C_\epsilon(1 + |k|)^{-1+\epsilon},$$

where $C, C_\epsilon > 0$ are independent in i, k and l.

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$$\bigcap_i \bigcup_{\substack{1 \leq p \leq q_i^{s/2} \\ p=1 \text{ or prime}}} \mathcal{N}_{q_i^{-1}}\left(\frac{\mathbb{Z}}{p}\right) \cap [-\frac{1}{2}, \frac{1}{2}]$$

with positive Fourier coefficients, associated with a measure $\nu \ll \mu$,

- $\frac{d\nu}{d\mu} \in L^\infty(\mu)$,
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and for each i and all integers $|k| > 2q_i, |l| < q_i/2$,

$$|\hat{\nu}(k + l)| \leq C\hat{\mu}(k) + C_\epsilon(1 + |k|)^{-1+\epsilon},$$

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Furthermore, μ does not admit Fourier frames. (Not via $\sum |\lambda|^{-s}$)

Construction on Diophantine approximation

All previous Kaufman-type constructions are actually supported on

$$\bigcap_{i} \bigcup_{\substack{2 \le p \le q_i^{s/2} \\ prime}} \mathcal{N}_{q_i^{-1}}\left(\frac{\mathbb{Z}}{p}\right) \cap [-\frac{1}{2}, \frac{1}{2}],$$

defined by the infinite product (with $\mathcal{P}_i$ a set of primes in $[2, q_i^{s/2}]$)

$$\prod_{i=1}^{\infty} F_i(x) := \prod_{i=1}^{\infty} \frac{1}{\#\mathcal{P}_i} \sum_{p \in \mathcal{P}_i} \sum_{v \in \mathbb{Z}} p^{-1} q_i \phi(q_i(x - \frac{v}{p})),$$

where $\phi \in C^2(-1, 1)$ used to be arbitrary (but not enough to us).

For the target measure μ, we take $\mathcal{P}_i^{\mu} = \{1\} \cup \{p \le q_i^{s/2}, prime\}$.

For the auxiliary measure ν, we take $\mathcal{P}_i^{\nu} = \{\frac{q_i^{s/2}}{\log q_i} \le p \le q_i^{s/2}, prime\}$.

The auxiliary function ϕ

We need an auxiliary function ϕ with the following properties:

1. $\phi \in C^2(\mathbb{R})$;
2. $\operatorname{supp} \phi \subset (-1, 1)$;
3. $\int \phi = 1$;
4. $\phi \geq 0$;
5. $\hat{\phi} \geq 0$;
6. $\hat{\phi}(\xi + r) \approx \hat{\phi}(\xi)$ uniformly in $\xi \in \mathbb{R}$ and $r \leq 1$. ($\phi \notin C_0^\infty$!)

In fact $\hat{\phi}(\xi) \approx (1 + |\xi|)^{-4}$ is sufficient for ⑥ and easier to check.

Now, fix $\phi_0 \in C_0^\infty(-1, 1)$, even, $\phi_0, \hat{\phi}_0 \geq 0$, and $\phi_0 \geq \frac{1}{2}$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Such a ϕ_0 exists by taking $\phi_0 = \varphi * \varphi$, where $\varphi \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$ is an arbitrary nonnegative even function satisfying $\varphi \geq 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Two ways to construct a desired ϕ

Explicit construction: let $\phi_1(x) = \chi_{[-1/2,1/2]}$, $\phi_2(x) = 2x|_{[-1/2,1/2]}$,

$$\phi(x) := A_1\phi_0(x) + A_2(\phi_1 * \phi_1 + \phi_2 * \phi_2^-) * (\phi_1 * \phi_1 + \phi_2 * \phi_2^-)(4x),$$

where $\phi_2^-(x) := \phi_2(-x)$, and $A_1, A_2 > 0$ are properly chosen. Then

$$\hat{\phi}(\xi) = A_1\hat{\phi}_0(\xi) + \frac{A_2}{4}\left((\frac{\sin \pi\xi/4}{\pi\xi/4})^2 + (\frac{\pi\xi/4 \cos \pi\xi/4 - \sin \pi\xi/4}{(\pi\xi/4)^2})^2 \right)^2,$$

strictly positive and $\lim_{|\xi|\to\infty}(\pi\xi)^4\hat{\phi}(\xi) = 4^3 A_2 > 0$.

Implicit construction (the Paley-Wiener theorem): take, for example,

$$F(z) = A_1\left(\frac{\pi z/4 - \sin \pi z/4}{z^3}\right)^2 + A_2\hat{\phi}_0(z),$$

with $A_1, A_2 > 0$ properly chosen. Then take the ϕ with $\hat{\phi} = F$.

Higher dimensions

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Q: any s-dimensional Salem measure in $[0,1]$ admit Fourier frames?

Thank you!